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Simone dos Santos
Universidade Federal do Piauí

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FREE VIBRATION ANALYSIS OF EULER-BERNOULLI BEAMS UNDER NON-CLASSICAL BOUNDARY CONDITIONS

João Fernandes da Silva, jfernandes.mecanica@gmail.com[1]
Lucas Allende Dias do Nascimento, allendezim@gmail.com[1]
Simone dos Santos Hoefel, simone.santos@ufpi.edu.br[1]

[1]Universidade Federal do Piauí, Campus Ministro Petrônio Portella, s/n - Ininga, Teresina-PI, 64049-550

Abstract: *The flexible beams carrying attachments and non-classical boundary conditions often appear in engineering structures, modal analysis of those structures is important and necessary in structural design. The analysis of vibrating beams with ends elastically restrained against rotation and translation or with ends carrying concentrated masses or rotational inertias is of great interest in a variety of practical cases. In this analysis, the governing differential equations of the beam, which is a partial differential equation with variable coefficients, and that of the mass-spring system, which is an ordinary differential equation, are found. The exact solution of the problem is then obtained using classical and non-classical boundary conditions and the eigenvalues and eigenfunctions are found. Finally, some cases with available results in the literature are presented and analyzed.*
Keywords: *Euler-Bernoulli beam, Non-classical boundary conditions, Natural frequencies*

1. INTRODUCTION

The interest in non-classical mechanical boundary conditions is originated in the study of stability of elastic structures, and a convenient reference for the various mechanical loadings considered until 1963 is the book by Bolotin (1963). The dynamic analysis of beams with ends elastically restrained against rotation and translation or with ends carrying concentrated masses or rotational inertia is a classical structural problem, which nowadays is becoming more and more important, even in mechanical engineering and in aeronautic engineering. Numerous authors have approached the analysis assuming that the beam is sufficiently slender to be considered as an Euler-Bernoulli beam, and trying to analytically solve the resulting fourth-order differential equation with variable coefficients. In exact method, difficulties arises in solving roots of the characteristic equation, except for very simple boundary conditions, that one has to go for numerical solution and in determining the normal modes of the system.

Grossi and Arenas (1996) employed both the classical Rayleigh-Ritz method and the optimized Rayleigh-Schmidt method to find the frequencies with varying width and height. A lot of numerical results were given, for various non-classical boundary conditions. Craver Jr. and Jampala (1993) examined the free vibration frequencies of a cantilever beam with variable cross-section and constraining springs. De Rosa and Auciello (1996) gave the exact free frequencies of a beam with linearly varying cross-section, in the presence of generic non-classical boundary conditions, so that all the usual boundary conditions can be treated as particular cases. Chun (1972) studied the free vibration of a Bernoulli-Euler beam hinged at one end by a rotational spring with constant spring stiffness and with the other end free. Wang and Lin (1996) utilized the Fourier series to investigate the dynamic analysis of beams having arbitrary boundary conditions. The Rayleigh-Ritz approach was used by Zhou and Cheung (2000) to find the first free vibration frequencies of three different tapered beams with various boundary conditions and truncation factors. Maurizi *et al.* (1976) studied the problem of free vibration of a uniform beam hinged at one end by a rotational spring and subjected to the restraining action of a translational spring at the other end using exact expression of trigonometric and hyperbolic functions. Maurizi *et al.* (1990) studied ends elastically restrained against rotation and translation in Timoshenko beams. Frequency equation of the cantilever beams carrying springs and masses by use the normal summation technique, was presented by Gürgöze (1984). Gürgöze (1998) also presented two formulations for the frequency equation of a clamped-free Euler-Bernoulli beam to which several spring-mass systems are attached in span. Recently, Liu and Gurram (2009) used his variational iteration method to calculate the natural frequencies and mode shapes of an Euler-Bernoulli beam under various supporting conditions.

This article focuses on the free vibration analysis of Euler-Bernoulli beams under non-classical boundary conditions. The governing differential equations of the beam are presented and the exact solution of the problem is then obtained using the pertinent boundary conditions. The eigenvalues and eigenfunctions of the problem which are frequencies and mode shapes of the system are derived for various properties of the system such as stiffness of springs and attached mass. Some numerical examples with available results in the literature are analyzed and its results discussed.

2. EULER-BERNOULLI BEAM THEORY

The partial differential equation of motion for free vibration of a Euler-Bernoulli beam is given by (Euler, 1773):

$$EI\frac{\partial^4 v(x,t)}{\partial x^4}+\rho A\frac{\partial^2 v(x,t)}{\partial t^2}=0, \tag{1}$$

where A is the cross-sectional area, I is the moment of inertia of the cross-sectional area, E is the modulus of elasticity, ρ is the mass per unit of volume and $v(x,t)$ is the transverse deflection at the axial location x and at the time t. Since the equation of the motion involves a second order derivative with respect to time and a fourth order derivative with respect to position, two initial equations and four boundary conditions are needed for finding a unique solution for $v(x,t)$. Equation of motion can be solved by using the methods of separation of variables and eigenfunctions expansion. The transverse deflection $v(x,t)$ are separated into two functions as follows:

$$v(x,t)\;=\;V(x)\,T(t), \tag{2}$$

where $V(x)$ is know as the normal mode or characteristic function of the beam. Therefore, the Eq. (1) is rewritten as (Han *et al.*, 1999):

$$c^2\frac{1}{V(x)}\frac{\partial^4 V(x)}{\partial x^4}=-\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2}=\omega^2, \tag{3}$$

knowing that $c=\sqrt{EI/\rho A}$ and ω is the natural frequency. Note that the left side of the Eq. (3) is relation to position and the right side to time and both can be splitted into two equations (Avcar, 2014):

$$\frac{d^4W(x)}{dx^4}-\beta^4W(x)=0, \tag{4}$$

$$\frac{d^2T(t)}{dt^2}+\omega^2T(t)=0, \tag{5}$$

where β is the eigenvalue of the normal mode $V(x)$ and

$$\beta=\sqrt{\frac{\omega}{c}}. \tag{6}$$

Solutions of $T(t)$ and $V(x)$ in Eq. (4) and Eq. (5), respectively, are given by:

$$T(t)\;=\;D\cos{(\omega t-\phi)}, \tag{7}$$
$$V(x)\;=\;C_1\sin(\beta x)+C_2\cos(\beta x)+C_3\sinh(\beta x)+C_4\cosh(\beta x). \tag{8}$$

C_1, C_2, C_3 and C_4 are constants which can be found by using the appropriate boundary conditions, D is a constant which can be obtained by using initial conditions and ϕ is the phase angle.

3. BOUNDARY CONDITIONS AND FREQUENCY EQUATIONS

Each one of the boundary conditions provides two equations necessary to determine the unknown constants C_1 to C_4 and the values of β in Eq. (8). They are usually classified in two groups: the classical and the non-classical boundary conditions. The classical boundary conditions are the most common in the vibration analysis in beams and are generally classified in free, clamped and supported ends and their equations are showed in Tab. 1. Non-classical boundary conditions are less common, but their importance are as the same as the classical conditions. These boundary conditions consists in points connected to masses, springs, rotational inertia or, even, dampers. All those conditions are presented in Tab. 2, where $a=-1$ and $b=1$ for the left end and $a=1$ and $b=-1$ for the right end. k_m is the stiffness of the linear spring, k_r is the stiffness of the torsional spring, c_m is the stiffness of the damper, c_r is the stiffness of the torsional damper, m_c is the value of the concentrated mass and the I_c is the value of the rotational inertia of the concentrated mass.

Table 1: Classical Boundary Conditions.

Boundary Condition	Equations
Free end	$EI\dfrac{\partial^2 v(x,t)}{\partial x^2} = 0$ and $EI\dfrac{\partial^3 v(x,t)}{\partial x^3} = 0$
Supported end	$v(x,t) = 0$ and $EI\dfrac{\partial^2 v(x,t)}{\partial x^2} = 0$
Clamped end	$v(x,t) = 0$ and $EI\dfrac{\partial v(x,t)}{\partial x} = 0$

Table 2: Non-classical Boundary Conditions.

Boundary Condition	Equations
Linear Spring	$EI\dfrac{\partial^2 v(x,t)}{\partial x^2} = 0$ and $EI\dfrac{\partial^3 v(x,t)}{\partial x^3} = ak_m v(x,t)$
Torsional Spring	$EI\dfrac{\partial^3 v(x,t)}{\partial x^3} = 0$ and $EI\dfrac{\partial^2 v(x,t)}{\partial x^2} = bk_r\dfrac{\partial v(x,t)}{\partial x}$
Concentrated Mass	$EI\dfrac{\partial^2 v(x,t)}{\partial x^2} = 0$ and $EI\dfrac{\partial^3 v(x,t)}{\partial x^3} = am_c\dfrac{\partial^2 v(x,t)}{\partial t^2}$
Rotational Inertia	$EI\dfrac{\partial^3 v(x,t)}{\partial x^3} = 0$ and $EI\dfrac{\partial^2 v(x,t)}{\partial x^2} = bI_c\dfrac{\partial^3 v(x,t)}{\partial x\,\partial t^2}$
Damper	$EI\dfrac{\partial^2 v(x,t)}{\partial x^2} = 0$ and $EI\dfrac{\partial^3 v(x,t)}{\partial x^3} = ac_m\dfrac{\partial v(x,t)}{\partial t}$
Torsional Damper	$EI\dfrac{\partial^3 v(x,t)}{\partial x^3} = 0$ and $EI\dfrac{\partial^2 v(x,t)}{\partial x^2} = bc_r\dfrac{\partial^2 v(x,t)}{\partial x\,\partial t}$

The application of appropriate boundary conditions presented in Tab. 2 into the Eq. (8) and its derivatives yields for each type of beam a set of four homogeneous linear algebraic equations in four constants C_1 to C_4. A non-trivial solution exists only if the determinant of the coefficients vanishes. The solving of the determinant yields the frequency equation or the characteristic equation for each type of beam. Table 3 show frequency equations for some beams under non-classical boundary conditions presented in Tab. 2, where:

$$
\begin{aligned}
Q_1 &= [\,cos(\,\beta_n\,l\,)\,cosh(\,\beta_n\,l\,) - 1\,], \\
Q_2 &= sin(\,\beta_n\,l\,)\,sinh(\,\beta_n\,l\,), \\
Q_3 &= sin(\,\beta_n\,l\,)\,cosh(\,\beta_n\,l), \\
Q_4 &= cos(\,\beta_n\,l\,)\,sinh(\,\beta_n\,l\,) \quad \text{and} \\
Q_5 &= cos(\,\beta_n\,l\,)\,cosh(\,\beta_n\,l\,).
\end{aligned}
$$

Table 3: Frequency equations for some beams under non-classical boundary conditions.

Boundary Condition	Frequency Equations
Linear Spring - Linear Spring	$(EI)^2\beta^6 Q_1 - 2k_m^2 Q_2 + 2EIk_m\beta^3[Q_3 - Q_4] = 0.$
Linear Spring - Concentrated Mass	$\rho AEI\beta^3 Q_1 + EIm_c\beta^4[Q_4 - Q_3] + k_m\rho A[Q_3 - Q_4] + 2k_m\beta Q_2 = 0.$
Linear Spring - Torsional Spring	$EI\beta k_m[Q_3 - Q_4] + EI\beta^3 k_r[Q_4 + Q_3] - 2k_m k_r Q_5 + (EI)^2\beta^4 Q_1 = 0.$
Torsional Spring - Torsional Spring	$2k_r^2 Q_2 + E^2I^2\beta^2 Q_1 + 2\beta k_r EI[Q_4 + Q_3] = 0.$

These frequency equations are highly transcendental and need to be solved numerically. In this paper was used the false position root finding method.

4. NUMERICAL EXAMPLES

This section presents numerical examples for Euler-Bernoulli beams subjected to three kind of boundary conditions. The same parameters values are considered for all examples and are given by $EI = 8000\,N.m^2$, $\rho A = 3000\,kg/m$, $L = 1\,m$, $m_c = 100\,kg$, $k_m = 1000\,N/m$, $k_r = 1000\,N.m/rad$ and the frequency equation are showed in Tab. 3 for each case analyzed.

4.1 Beam connected to linear springs at both ends

Consider a beam connected to linear springs at both ends shown in Fig. 1, the natural frequencies values and the eigenvalues obtained from the frequency equation are presented in Tab. 4.

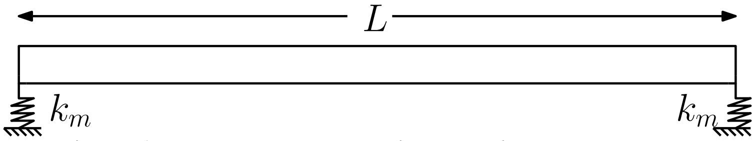

Figure 1: Beam connected to linear springs at both ends.

Table 4: Natural frequencies values ω and eingenvalues β_n of a Euler-Bernoulli beam connected to linear springs at both ends.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
Eigenvalues β_n	$4,7324$	$7,8537$	$10,9958$	$14,1373$	$17,2788$	$20,4204$	$23,5620$
Natural frequencies ω_n	$36,5719$	$100,7245$	$197,4412$	$326,3732$	$487,5419$	$680,9451$	$906,5825$

The normal mode $V(x)$ and the slope $\theta(x)$, which is the derivative of $V(x)$, are as follow:

$$V(x) = \sigma_1[cos(\beta x) + cosh(\beta x)] + \sigma_2 sin(\beta x) + sinh(\beta x), \tag{9}$$

$$\theta(x) = \beta\{\sigma_1[-sin(\beta x) + sinh(\beta x)] + \sigma_2 cos(\beta x) + cosh(\beta x)\}, \tag{10}$$

where

$$\sigma_1 = \frac{EI\beta^3[sinh(\beta_n l) - sin(\beta_n l)]}{-EI\beta^3 cosh(\beta_n l) + 2k_m sin(\beta_n l) + EI\beta^3 cos(\beta_n l)}$$

and

$$\sigma_2 = \frac{2k_m sinh(\beta_n l) - EI\beta^3 cos(\beta_n l) + EI\beta^3 cosh(\beta_n l)}{-EI\beta^3 cosh(\beta_n l) + 2k_m sin(\beta_n l) + EI\beta^3 cos(\beta_n l)}.$$

Figure 2 showed mode shape of the first four eigenvalues and Fig. 3 showed the plots of slope. We can observed that the major deflection always occurs at the ends of the beam. Note that the modal shapes and graphs of slope of this beam are very similar to those of a free-free beam, as showed in Fig. 4 and Fig. 5. Also, if the stiffness k_m is equal to zero, the resulting frequency equation is one for a free-free beam as follows

$$cos(\beta_n l)\, cosh(\beta_n l) - 1 = 0. \tag{11}$$

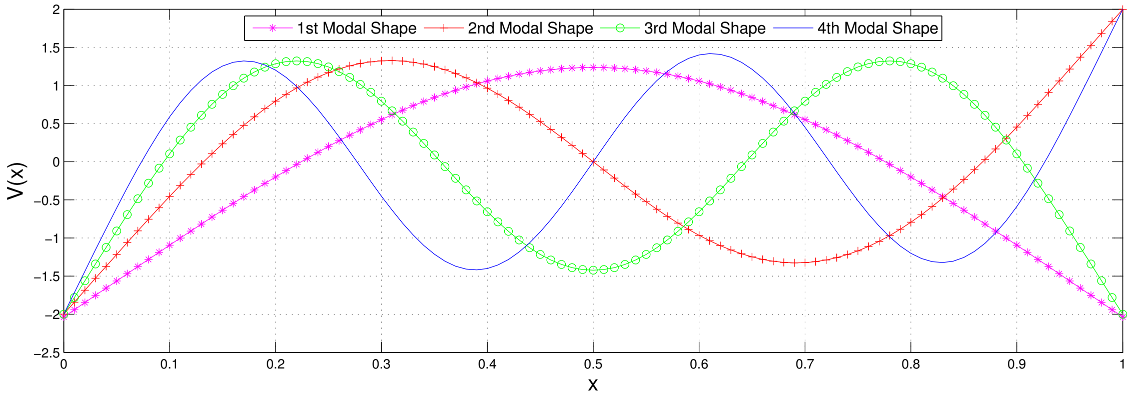

Figure 2: Modal Shapes for a Euler-Bernoulli beam connected to linear springs at both ends.

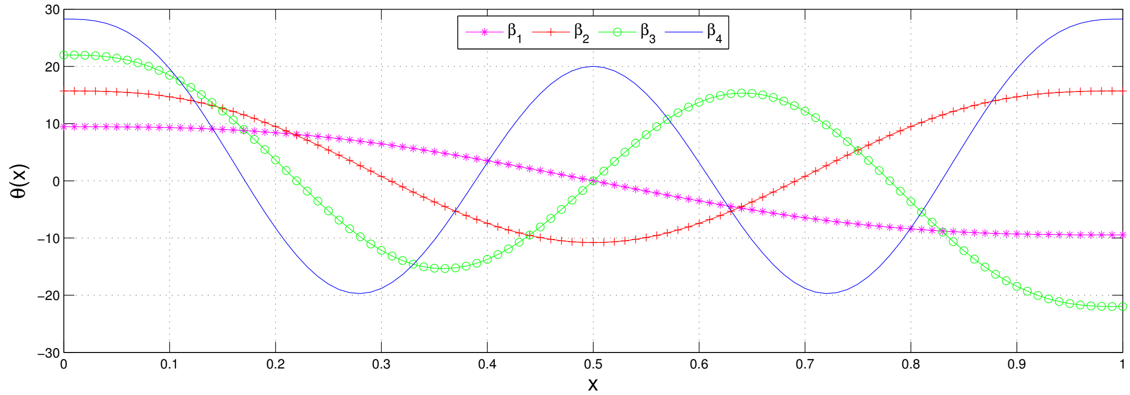

Figure 3: Slope of Euler-Bernoulli beam connected to linear springs at both ends associate to the first four eigenvalues β_n.

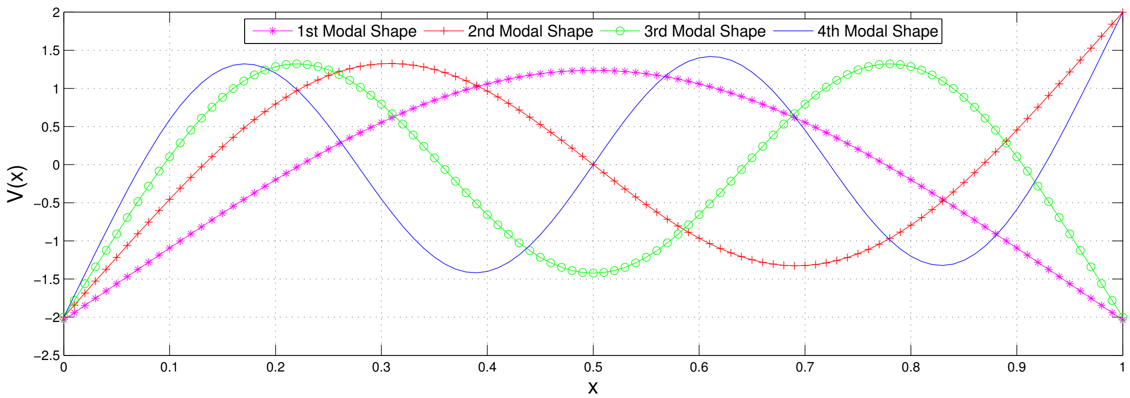

Figure 4: Modal Shapes for a free-free Euler-Bernoulli beam.

Furthermore, the natural frequency values found are very closer from the values for a free-free beam, which are presented in Tab. 5. For a free-free beam, the normal mode V(x) and the slope $\theta(x)$ are:

$$V(x) = \sigma[cos(\beta x) + cosh(\beta x)] + sin(\beta x) + sinh(\beta x), \tag{12}$$

$$\theta(x) = \beta\{\sigma[-sin(\beta x) + sinh(\beta x)] + cos(\beta x) + cosh(\beta x)\}, \tag{13}$$

where

$$\sigma = \frac{sin(\beta_n l) - sinh(\beta_n l)}{cosh(\beta_n l) - cos(\beta_n l)}.$$

4.2 Beam connected to a linear spring at left end and to a concentrated mass at right end

Now, a beam connected to a linear spring at left end and to a concentrated mass at right end shown in Fig. 6 is analyzed. The normal mode $V(x)$ and the slope $\theta(x)$ are:

$$V(x) = \sigma_1[cos(\beta x) + cosh(\beta x)] + \sigma_2 sin(\beta x) + sinh(\beta x), \tag{14}$$

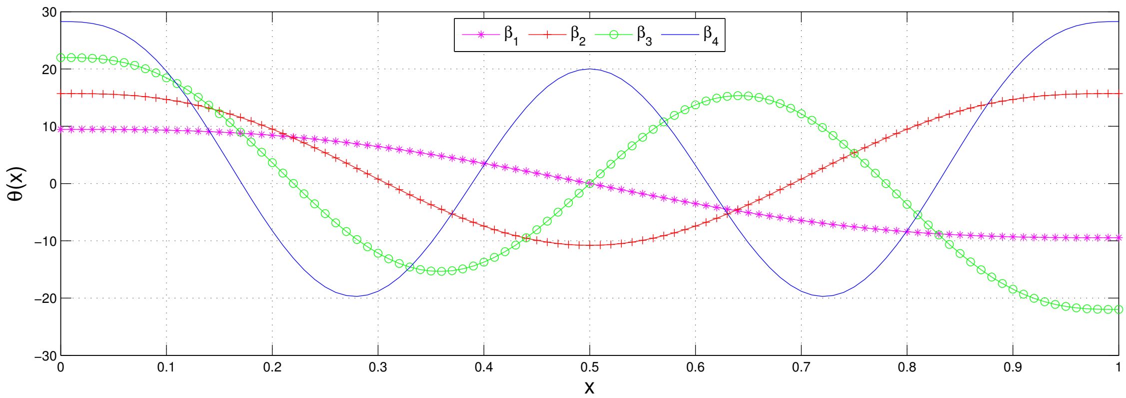

Figure 5: Slope of free-free Euler-Bernoulli beam associate to the first four eigenvalues β_n.

Table 5: Natural frequencies values ω and eigenvalues β_n of a free-free Euler-Bernoulli beam.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
Eigenvalues β_n	$4,7300$	$7,8532$	$10,9956$	$14,1372$	$17,2788$	$20,4204$	$23,5619$
Natural frequencies ω_n	$36,5354$	$100,7113$	$197,4344$	$326,3691$	$487,5391$	$680,9431$	$906,5811$

$$\theta(x) = \beta\{\sigma_1[-sin(\beta x) + sinh(\beta x)] + \sigma_2 cos(\beta x) + cosh(\beta x)\}, \tag{15}$$

where

$$\sigma_1 = \frac{EI\beta^3[sinh(\beta_n l) - sin(\beta_n l)]}{-EI\beta^3 cosh(\beta_n l) + 2k_m sin(\beta_n l) + EI\beta^3 cos(\beta_n l)}$$

and

$$\sigma_2 = \frac{2k_m sinh(\beta_n l) - EI\beta^3 cos(\beta_n l) + EI\beta^3 cosh(\beta_n l)}{-EI\beta^3 cosh(\beta_n l) + 2k_m sin(\beta_n l) + EI\beta^3 cos(\beta_n l)}.$$

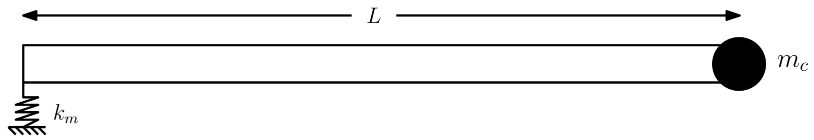

Figure 6: Beam connected to a linear spring at left both end and to a concentrated mass at right end.

The natural frequencies values ω and the eigenvalues β are presented in Tab. 6. As the previous beam, the modal shapes in Fig. 7 and the slopes in Fig. 8 the are similar to those of a free-free beam, except for the deflection at left end be greater than at the right end. Besides, considering m_c and k_m equals to zero, the frequency equation becomes equal to frequency equation of free-free beam.

Table 6: Natural frequencies values ω and eigenvalues β_n of a Euler-Bernoulli beam connected to spring at left end and to a concentrated mass at right end.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
Eigenvalues β_n	$4,5988$	$7,6530$	$10,7380$	$13,8315$	$16,9325$	$20,0395$	$23,1512$
Natural frequencies ω_n	$34,5368$	$95,6418$	$188,2911$	$312,4104$	$468,1967$	$655,7769$	$875,2458$

4.3 Beam connected to a linear spring at left end and to a torsional spring at right end

In this example, a beam connected to a linear spring at left end and to a torsional spring at right end is considered, as showed in the Fig. 9. The natural frequencies values and the eigenvalues are presented in Tab. 7 and the normal mode $V(x)$ and the slope $\theta(x)$ are:

$$V(x) = \sigma_1[-cos(\beta x) - cosh(\beta x)] + \sigma_2 sin(\beta x) + sinh(\beta x), \tag{16}$$

$$\theta(x) = \beta\{\sigma_1[sin(\beta x) - sinh(\beta x)] + \sigma_2 cos(\beta x) + cosh(\beta x)\}, \tag{17}$$

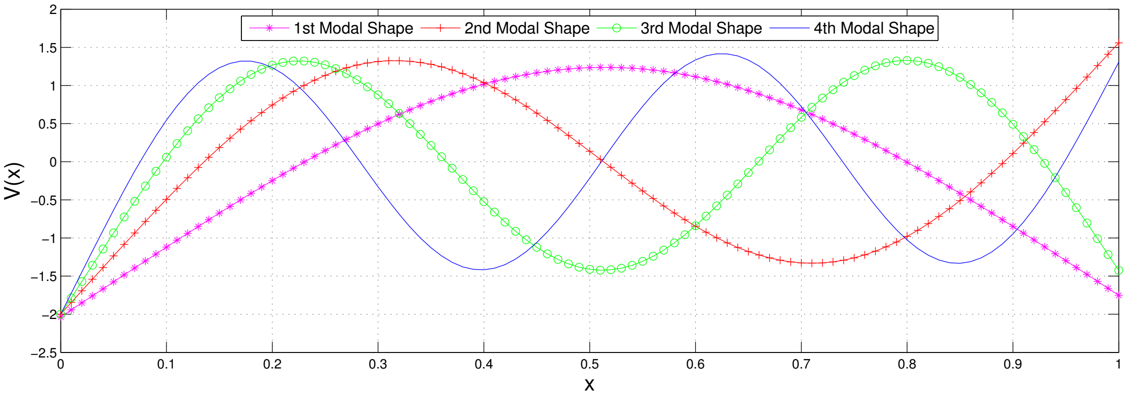

Figure 7: Modal Shapes for a Euler-Bernoulli beam connected to a linear spring at left end and to a concentrated mass at right end.

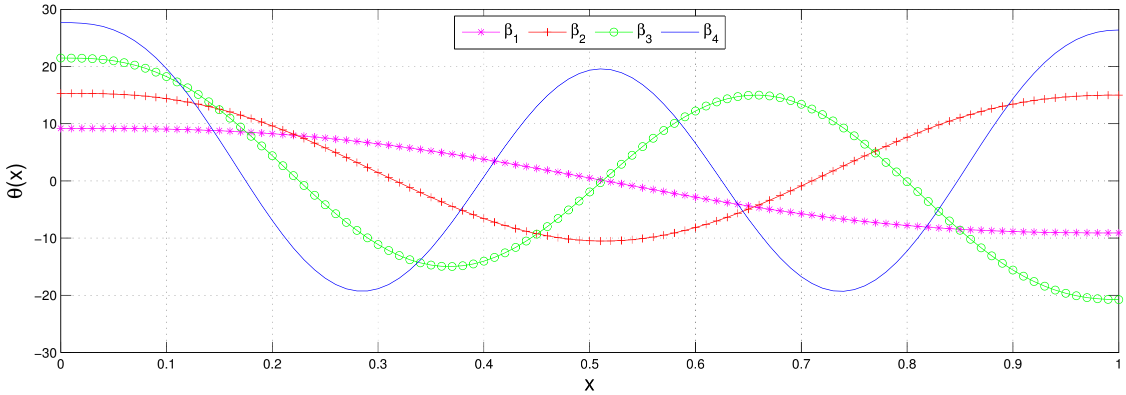

Figure 8: Slope of Euler-Bernoulli beam connected to a linear spring at left end and to a concentrated mass at right end associate to the first four eigenvalues β_n.

where

$$\sigma_1 = \frac{EI\beta^3[cos(\beta_n l) - cosh(\beta_n l)]}{2k_m cos(\beta_n l) - EI\beta^3 sinh(\beta_n l) - EI\beta^3 sin(\beta_n l)}$$

and

$$\sigma_2 = \frac{2k_m cosh(\beta_n l) - EI\beta^3 sinh(\beta_n l) - EI\beta^3 sin(\beta_n l)}{2k_m cos(\beta_n l) - EI\beta^3 sinh(\beta_n l) - EI\beta^3 sin(\beta_n l)}.$$

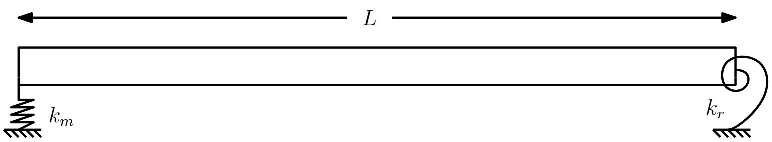

Figure 9: Beam connected to a linear spring at left both end and to a torsional spring at right end.

Modal shapes presented in Fig. 10 and the plots of slopes in Fig. 11 are still very similar to those of a free-free beam and to those of the previous beams. Furthermore, if k_m and k_r are zero, the result is similar to the frequency equation for a free-free beam.

Table 7: Natural frequencies values ω and eingenvalues β_n of a Euler-Bernoulli beam connected to a linear spring at left end and to a torsional spring at right end.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
Eigenvalues β_n	$4,7559$	$7,8691$	$11,0069$	$14,1460$	$17,2860$	$20,4264$	$23,5672$
Natural frequencies ω_n	$36,9365$	$101,1199$	$197,8412$	$326,7757$	$487,9457$	$681,3498$	$906,9878$

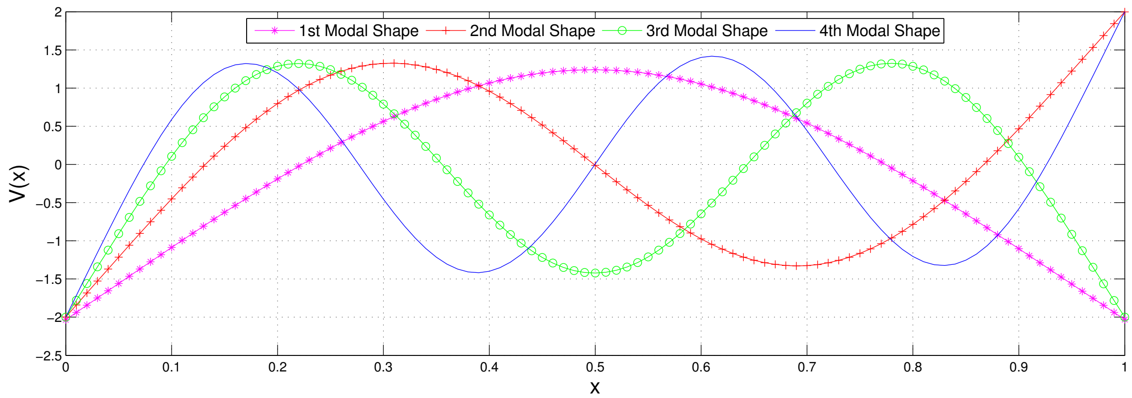

Figure 10: Modal Shapes for a Euler-Bernoulli beam connected to a linear spring at left end and to a torsional spring at right end.

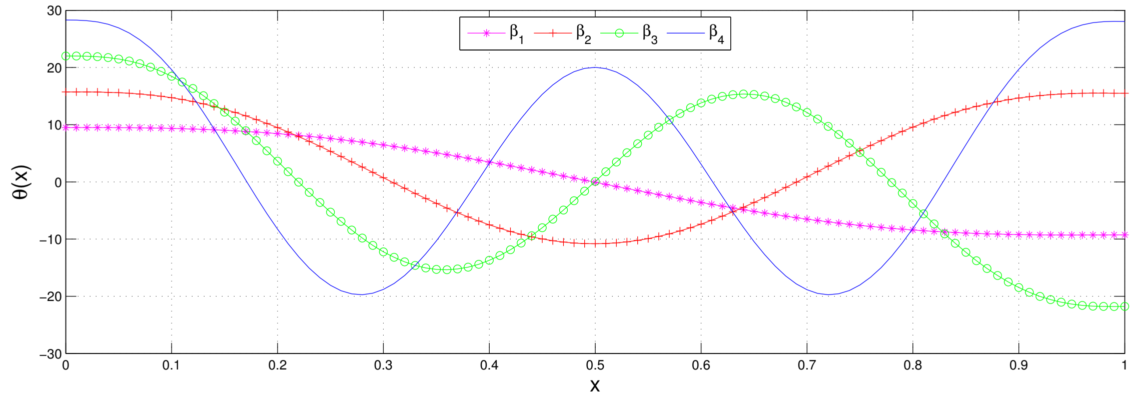

Figure 11: Slope of Euler-Bernoulli beam connected to a linear spring at left end and to a torsional spring at right end associate to the first four eigenvalues β_n.

4.4 Beam connected to torsional springs at both ends

Finally, consider a beam connected to a torsional spring at left end and to a torsional spring at right end as showed in Fig. 12 and with normal mode $V(x)$ and the slope $\theta(x)$ given by:

$$V(x) = \sigma_1[sin(\beta x) + sinh(\beta x)] + \sigma_2 cos(\beta x) + cosh(\beta x), \tag{18}$$

$$\theta(x) = \beta\{\sigma_1[cos(\beta x) + cosh(\beta x)] - \sigma_2 sin(\beta x) + senh(\beta x)\}, \tag{19}$$

where

$$\sigma_1 = \frac{EI\beta[sinh(\beta_n l) + sin(\beta_n l)]}{EI\beta[cos(\beta_n l) - cosh(\beta_n l)] + 2\beta k_r sin(\beta_n l)}$$

and

$$\sigma_2 = \frac{EI\beta[cos(\beta_n l) - cosh(\beta_n l)] - 2\beta k_r sinh(\beta_n l)}{EI\beta[cos(\beta_n l) - cosh(\beta_n l)] + 2\beta k_r sin(\beta_n l)}.$$

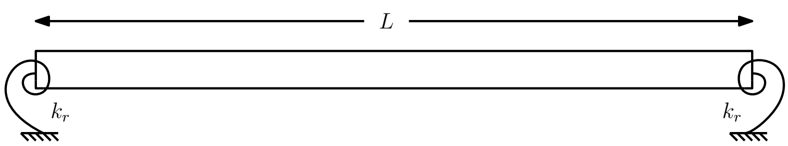

Figure 12: Beam connected to torsional springs at both ends.

According Tab. 8, the highest values of the natural frequencies values ω and the eigenvalues β were found for the beam connected to torsional springs. For this beam, the modal shapes showed in the Fig. 13 and the graphs of slope in Fig. 14 are the inverse for those of each one of the preceding beams and for a free-free beam, but they still have a similarity. Besides, the frequency equation for a free-free beam is obtained if the value of k_r is zero.

Table 8: Natural frequencies values ω and eingenvalues β_n of a Euler-Bernoulli beam connected to torsional springs at both ends.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
Eigenvalues β_n	$4,7793$	$7,8845$	$11,0180$	$14,15470$	$17,2931$	$20,4325$	$23,5725$
Natural frequencies ω_n	$37,3004$	$101,5146$	$198,2408$	$327,1779$	$488,3494$	$681,7544$	$907,3931$

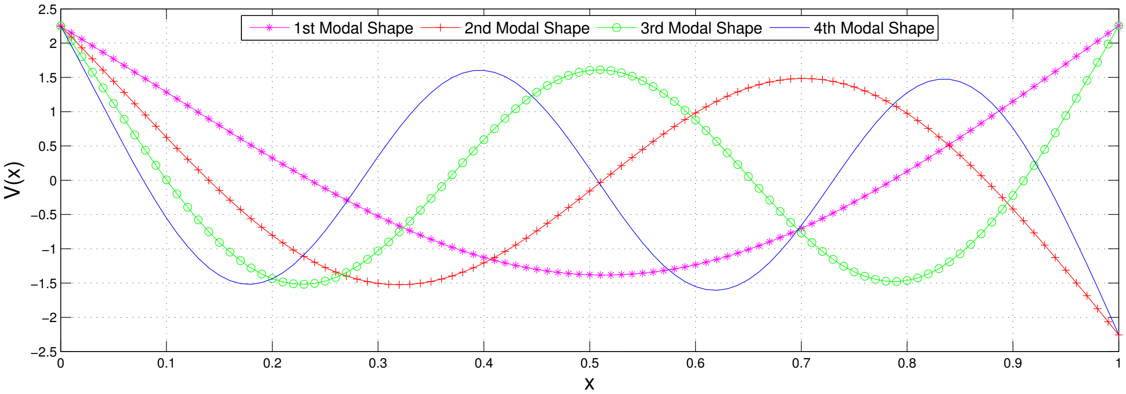

Figure 13: Modal Shapes for a Euler-Bernoulli beam connected to torsional springs at both ends.

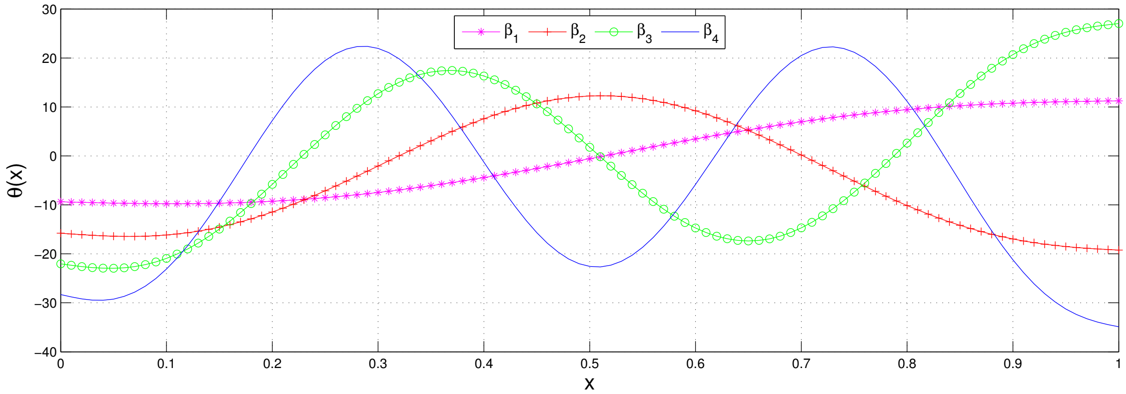

Figure 14: Slope of Euler-Bernoulli beam connected to torsional springs at both ends associate to the first four eigenvalues β_n.

5. CONCLUSION

In this paper, free vibration of beams considering different boundary conditions are analyzed. The natural frequencies and the eigenvalues for Euler-Bernoulli beams are obtained and their modal shapes are showed in graphs. The eigenvalues and the eigenfunctions, for each case analyzed, are very similar to the available results in the literature, like the studies conducted by Morelatto (2000).

It was noted that the major deflection and slope of the beams in study always happens at their ends. That occurs because the spring-mass-beam system presents an initial deflection in order to present an static equilibrium. Since that does not have any excitation after the static equilibrium, the springs or masses will not deflect anymore. Meanwhile, in free vibration, the entire beam is vibrating trying to restore to the position of equilibrium, always finding a maximum deflection and slopes at the ends of the beams. For this reason, the behavior of these beams are similar to a free-free beam, even their frequency equation can become that of a free-free beam, if the values for the stiffness of the springs and for the masses are equals to zero.

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