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Simone dos Santos
Universidade Federal do Piauí

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Lucas Silva Soares

Wallison Kennedy da Silva Bezerra

Simone dos Santos Hoefel

soares.l.s.2373@gmail.com

wallisonkennedy1@hotmail.com

simone.santos@ufpi.edu.br

Laboratório de Mecânica Computacional

Universidade Federal do Piauí, Campus Universitário Ministro Petrônio Portela - Bairro Ininga, CEP: 64049-550, Teresina, Piauí, Brasil.

Abstract. *Free vibration analysis of beams on the elastic foundation is necessary for an optimal design in many engineering applications. This paper analyzes the effect of Winkler and Pasternak foundations in the natural frequencies of a Timoshenko beam. For this purpose, the natural frequencies are obtained by solving the partial differential equation governing the problem. A finite element is developed using cubic and quadratic polynomials for transverse displacement and slope, respectively, for a two-node beam element with two degrees of freedom per node. The finite element and the analytic solutions are compared and discussed with some numerical examples. The results presented a high accuracy and reliability for the beam-foundation iteration problem. The presence of the Pasternak foundation increases the natural frequencies of the beam. The growth of the frequencies on Pasternak foundation is mostly due to the shear layer stiffness, and it reduces the influence of the elastic stiffness. The use of the Euler-Bernoulli beam on Winkler foundation instead of the Timoshenko beam on Pasternak foundation presents a significant inaccuracy, except for specific values of rotational inertia.*

Keywords: *Free Vibration Analysis, Timoshenko Beam Theory, Winkler Foundation, Pasternak Foundation, Finite Element Analysis*

1 INTRODUCTION

Beams are structural elements widely used in mechanical, civil and geotechnical engineering as it is capable of simulating many structures behavior. Such structures are often used or modeled on an elastic foundation for isolation purpose, to know the behavior of buildings on soil and railways applications. Hence, the optimal design of these structures lies in the knowledge of its dynamical characteristics. Therefore, the vibration analysis of beams on elastic foundation represents an important study for engineering applications.

The search for a model which can account for the realistic response of a system is the aim of most researchers in soil-structures interaction field. However, the difficulty in determining the input parameters due the various type of soils and the interaction with the structure makes a rigorous model a unpractical task for most of the engineering applications. Thus, the assumption of a linear elastic, homogeneous and isotropic behavior gives a reliable information for practical problems. From theses hypotheses, is possible to use two approaches: the continuous medium model and the mechanical models (Selvadurai, 1979).

The continuous medium model attempts to simulate the elastic behavior of the soil media and its cohesion with a three-dimensional continuous elastic solid. However, this model presented a less precise response in regions away from the loaded region and a difficulty to obtain the analytic solution even with simplifying assumption (Dutta and Roy, 2002).

The mechanical models have presented an alternative to the mathematical complexity of the continuous medium model. Although less precise, a mechanical model is simpler and easier to use and, presents a good response for most engineering applications (Hetenyi, 1946). The forerunner of the mechanical models was the Winkler foundation. This idealization takes into account the resistance against vertical deformation, in which the foundation is modeled as a series of closely spaced, independent and linear elastic vertical springs (Winkler, 1867). However, as the springs are independent, the Winkler foundation presents no cohesion, which the displacement is only localized immediately under the applied load. This characteristic represents the major drawback in this model (Dutta and Roy, 2002).

Thus, some two-parameter models were developed to include the effect of continuity and cohesion of the soil. A two parameter foundation is a model in which the effects of the interaction between springs and cohesion of the soil are taken into account (Selvadurai, 1979). Various types of two-parameter foundation models, such as those of Pasternak, Filonenko-Borodich, Hetenyi, Vlasov, and Reissner have been presented as a modified version of Winkler foundation to account continuity through interaction amongst the spring elements by some structural elements (Dutta and Roy, 2002). Kerr (1964) showed that the Pasternak foundation is the most natural extension of the Winkler model for homogeneous foundations. The Pasternak model includes the soil cohesion by a shear layer of incompressible vertical elements that resist only to transverse shear is attached to the end of the springs (Pasternak, 1954).

A series of studies have concerned the dynamic analysis of beams on elastic foundations. Wang and Stephens (1977) studied the natural frequencies of a Timoshenko beam on Pasternak foundation and derived the frequency equations to various boundary conditions. Yokoyama (1987) presented a finite element method to analyze the free vibration and transient response of a Timoshenko beam on Winkler and Pasternak foundations. De Rosa (1995) showed the free vibration of a Timoshenko beam on two proposed types of generalized elastic foundations. A finite element analysis was also performed by Thambiratnam and Zhuge (1996) and applied to particular cases of stepped beams on elastic foundation, beam on stepped foundation and continuous beams. Yokoyama (1996) showed a finite element vibration analysis of Euler-Bernoulli and Timoshenko beam-columns on a two-parameter elastic foundation. El-Mously (1999) derived the fundamental natural frequencies of vibration of finite Timoshenko beams on Pasternak foundation by Rayleigh's principle. Chen et al. (2004) studied a mixed method that combines the state space method and the differential quadrature method to the free vibration of Euler-Bernoulli beams on Pasternak foundation and discussed the influence of Poisson's ratio and foundation parameters. Lee et al. (2014) studied the flexural-torsional free vibrations of finite uniform beams resting on finite Pasternak foundation. Ghannadiasl and Mofid (2015) presented the exact solution to free vibration of elastically restrained Timoshenko beam on an arbitrary variable elastic foundation using Green functions.

Thus, this paper presents a finite element analysis of the natural frequencies of Timoshenko beam on Winkler and Pasternak foundations. The results obtained by finite element is compared with the analytic solution to validate the accuracy of the method. The study compares the Timoshenko beam on Winkler and Pasternak foundations to determine the difference in adopting each theory. Also, the influence of rotatory inertia, transverse shear deformation and foundation parameters on the natural frequencies is discussed.

2 CLASSICAL THEORY

Figure 1 shows a scheme of a uniform Timoshenko beam on a Pasternak foundation. The adopted Timoshenko beam theory (TBT) is a major improvement for non-slender beams and for higher frequency responses where shear or rotatory effects are not negligible (Timoshenko, 1921; Soares and Hoefel, 2015). According to Kerr (1964), the Pasternak foundation is one of the most used models and constitutes a more accurate model of the soil medium when compared to Winkler model.

The potential energy of the beam-foundation system is given by (Yokoyama, 1987):

$$\begin{aligned} U \;=\; & \frac{1}{2}\int_0^L EI\left(\frac{\partial\psi(x,t)}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^L \kappa AG\left(\psi(x,t) - \frac{\partial v(x,t)}{\partial x}\right)^2 dx \;+ \\ & \frac{1}{2}\int_0^L k_f\,(\nu)^2\,dx + \frac{1}{2}\int_0^L G_p\left(\frac{\partial \nu}{\partial x}\right)^2 dx, \end{aligned} \tag{1}$$

where L is the length of beam, A, the cross-sectional area, I, the moment of inertia of cross section, E, the modulus of elasticity, G the modulus of rigidity, κ is the shape factor or shear coefficient, k_f the foundation stiffness coefficient, G_p the foundation shear coefficient, $\nu(x,t)$ is the transverse deflection and $\psi(x,t)$ is the beam slope due to bending at the axial location x and time t.

The kinetic energy is expressed as:

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial\nu(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L \rho I\left(\frac{\partial\psi(x,t)}{\partial t}\right)^2 dx, \tag{2}$$

where ρ is the mass per unit volume. The equation of motion can be obtained using Hamilton's principle:

$$\int_{t_1}^{t_2} \delta(T-U)dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0, \tag{3}$$

where δW_{nc} is the virtual work due non conservative forces, t_1 and t_2 are times at which the configuration of the system is known and δ is the symbol denoting virtual change. Substituting Eqs. (1) and (2) on the Eq. (3), after some manipulations, one can obtain two coupled differential equations for free vibration response:

$$\rho A \frac{\partial^2 \nu(x,t)}{\partial t^2} + \kappa AG \left(\frac{\partial \psi(x,t)}{\partial x} - \frac{\partial^2 \nu(x,t)}{\partial x^2} \right) + k_f \nu(x,t) - G_p \frac{\partial^2 \nu(x,t)}{\partial x^2} = 0 \tag{4}$$

and

$$EI \frac{\partial^2 \psi(x,t)}{\partial x^2} - \rho I \frac{\partial^2 \psi(x,t)}{\partial t^2} - \kappa AG \left(\psi(x,t) - \frac{\partial \nu(x,t)}{\partial x} \right) = 0. \tag{5}$$

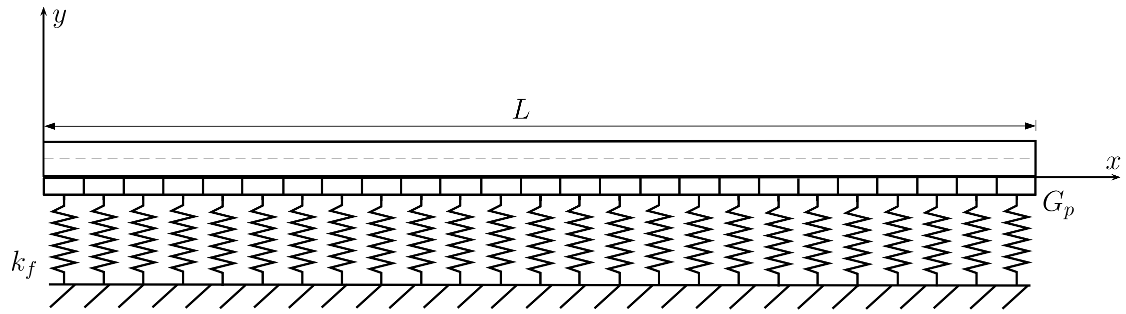

Figure 1: A beam on a Pasternak foundation.

Equations (4) and (5) leads to uncoupled differential equations for the beam deflection and slope:

$$EI\left(1+\frac{G_p}{\kappa AG}\right)\frac{\partial^4 \nu}{\partial x^4} + \left(\rho A + \frac{k_f \rho I}{\kappa AG}\right)\frac{\partial^2 \nu}{\partial t^2} - \left(\frac{EIk_f}{\kappa AG} + G_p\right)\frac{\partial^2 \nu}{\partial x^2} - \left[\rho I\left(1+\frac{E}{\kappa G}\right) + \frac{G_p \rho I}{\kappa AG}\right]\frac{\partial^4 \nu}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G}\frac{\partial^4 \nu}{\partial t^4} + k_f \nu = 0 \tag{6}$$

and

$$EI\left(1+\frac{G_p}{\kappa AG}\right)\frac{\partial^4 \psi}{\partial x^4} + \left(\rho A + \frac{k_f \rho I}{\kappa AG}\right)\frac{\partial^2 \psi}{\partial t^2} - \left(\frac{EIk_f}{\kappa AG} + G_p\right)\frac{\partial^2 \psi}{\partial x^2} - \left[\rho I\left(1+\frac{E}{\kappa G}\right) + \frac{G_p \rho I}{\kappa AG}\right]\frac{\partial^4 \psi}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{\kappa G}\frac{\partial^4 \psi}{\partial t^4} + k_f \psi = 0. \tag{7}$$

Assuming that the beam is excited harmonically with an angular frequency ω and:

$$\nu(x,t) = V(x)\cdot e^{i\omega t}, \quad \psi(x,t) = \Psi(x)\cdot e^{i\omega t},$$

$$\xi = x/L, \quad b^2 = \frac{\rho A L^4}{EI}\omega^2, \tag{8}$$

where i = $\sqrt{-1}$, ξ is the non-dimensional length of the beam and $V(x)$ and $\Psi(x)$ are the normal functions of $\nu(x)$ and $\psi(x)$, respectively. Substituting the relations presented in Eq. 8 into Eq. 6 and Eq. 7 and, omitting the common term $e^{i\omega t}$ we obtain (Wang and Stephens, 1977; De Rosa, 1995):

$$\frac{d^4 V(\xi)}{d\xi^4} + \gamma \frac{d^2 V(\xi)}{d\xi^2} + \zeta V(\xi) = 0, \tag{9}$$

$$\frac{d^4\Psi(\xi)}{d\xi^4} + \gamma\frac{d^2\Psi(\xi)}{d\xi^2} + \zeta\Psi(\xi) = 0, \tag{10}$$

where

$$\gamma = \frac{b^2\left(r^2+s^2\right) - s^2e^2 + p^2\left(b^2r^2s^2-1\right)}{1+s^2p^2}, \quad \zeta = \frac{\left(b^2-e^2\right)\left(b^2r^2s^2-1\right)}{1+s^2p^2}$$

and r, s, e and p are the coefficients related with the effect of rotary inertia, shear deformation, elastic and shear layer stiffness, respectively, given by:

$$r^2 = \frac{I}{AL^2},\ s^2 = \frac{EI}{\kappa GAL^2},\ e^2 = \frac{k_f L^4}{EI},\ p^2 = \frac{G_p L^2}{EI}. \tag{11}$$

In order to solve the O.D.E. of Eqs. 9 and 10, two conditions must be considered. This conditions represents different solution expressions. For the first case:

$$\zeta < 0,\ \text{which leads to: } b < \frac{1}{rs} \text{ and } b > e \ \text{ or } \ b > \frac{1}{rs} \text{ and } b < e. \tag{12}$$

This condition results in the solutions to be expressed in trigonometric and hyperbolic functions:

$$V(\xi) = C_1\, cosh(\alpha_1\xi) + C_2\, sinh(\alpha_1\xi) + C_3\, cos(\beta\xi) + C_4\, sin(\beta\xi), \tag{13}$$

$$\Psi(\xi) = C_1'\, sinh(\alpha_1\xi) + C_2'\, cosh(\alpha_1\xi) + C_3'\, sin(\beta\xi) + C_4'\, cos(\beta\xi), \tag{14}$$

where:

$$\alpha_1 = \frac{\sqrt{2}}{2}\sqrt{-\gamma + \sqrt{\gamma^2 - 4\zeta}}, \tag{15}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\ \gamma + \sqrt{\gamma^2 - 4\zeta}}, \tag{16}$$

and C and C' are constants.

The second case gives:

$$\zeta > 0,\ \text{which leads to: } b > \frac{1}{rs} \text{ and } b > e \ \text{ or } \ b < \frac{1}{rs} \text{ and } b < e. \tag{17}$$

As a result, the solution is expressed only in trigonometric functions:

$$V(\xi) = \bar{C}_1\, cos(\alpha_2\xi) + \bar{C}_2\, sin(\alpha_2\xi) + \bar{C}_3\, cos(\beta\xi) + \bar{C}_4\, sin(\beta\xi), \tag{18}$$

$$\Psi(\xi) = \bar{C}_1'\, sin(\alpha_2\xi) + \bar{C}_2'\, cos(\alpha_2\xi) + \bar{C}_3'\, sin(\beta\xi) + \bar{C}_4'\, cos(\beta\xi), \tag{19}$$

where:

$$\alpha_2 = \frac{\sqrt{2}}{2}\sqrt{\gamma - \sqrt{\gamma^2 - 4\zeta}}, \tag{20}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\gamma + \sqrt{\gamma^2 - 4\zeta}}, \tag{21}$$

and $\bar{C}$ and $\bar{C}'$ are constants.

Equations 12 and 17 presents the conditions to distingue two behaviours of the Timoshenko beam. Azevedo et al. (2016a, 2016b) studied this phenomenon, the so-called second spectrum, and showed that this occurs because of the difference of phase between the bending and shear deformation. This paper concerns the natural frequencies for the first spectrum.

Equations 9 and 10 shows that the beam-foundation theory represents a generalization of the beam theory. Disregarding the parameters e and p, the solution regress to the solution of a beam without foundation. Also, the Pasternak foundation theory is a higher generalization as it includes the solution for Winkler when $p = 0$.

3 FINITE ELEMENT FORMULATION

Consider a uniform Timoshenko beam element on Pasternak Foundation as shown in Fig. 2. The beam element consists of two nodes and each node has two degrees of freedom: V, the total deflection, and Ψ, the slope due to bending.

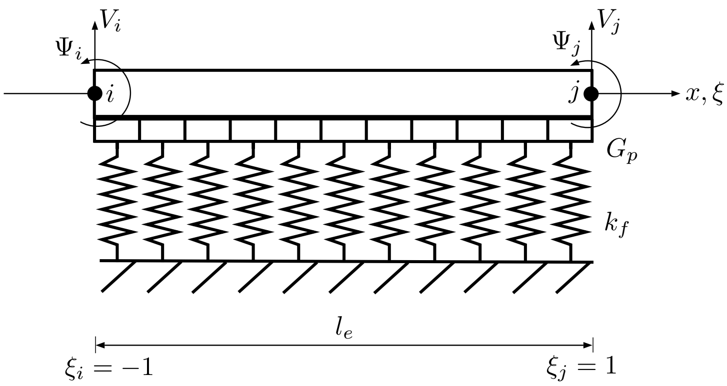

Figure 2: Beam on Pasternak foundation element

Solving the homogeneous form of Timoshenko beam static equations, one can obtain a cubic and quadratic displacement functions as follows (Yokoyama, 1987):

$$V_i(\xi) = \sum_{i=0}^{3} \lambda_i \xi^i \quad \text{and} \quad \Psi_i(\xi) = \sum_{i=0}^{2} \overline{\lambda}_i \xi^i. \tag{22}$$

where λ_i and $\overline{\lambda}_i$ are constants.

Using the non-dimension coordinate, ξ, and element length, l_e, the matrix form of the displacement V and total slope Ψ can be written as:

$$V = [\,\mathbf{N}(\xi)\,]\{\,\mathbf{v}\,\}_e \quad \text{and} \quad \Psi = [\,\overline{\mathbf{N}}(\xi)\,]\{\,\mathbf{v}\,\}_e, \tag{23}$$

where $[\,\mathbf{N}(\xi)\,]$ and $[\,\overline{\mathbf{N}}(\xi)\,]$ are the shape functions and $\{\,\mathbf{v}\,\}_e$ is the vector of nodal coordinates. The subscript e represents expressions for a single element.

Therefore, the shape functions in Eq. 23 can be expressed as:

$$\mathbf{N_i}(\xi) = \frac{1}{4(1+3\beta)} \begin{bmatrix} 2(3\beta+1) - 3(\beta+1)\xi + \xi^3 \\ (l_e/2)\,[3\beta + 1 - \xi - (3\beta+1)\xi^2 + \xi^3] \\ 2(3\beta+1) + 3(2\beta+1)\xi - \xi^3 \\ (l_e/2)\,[-3\beta - 1 - \xi + (3\beta+1)\xi^2 + \xi^3] \end{bmatrix}^T, \tag{24}$$

and

$$\overline{\mathbf{N}}_i(\xi) = \frac{1}{4(1+3\beta)} \begin{bmatrix} (l_e/2)\,(3\xi^2-3) \\ -1-2(3\beta+1)\xi+6\beta+3\xi^2 \\ (l_e/2)\,(3-3\xi^2) \\ -1+2(3\beta+1)\xi+6\beta+3\xi^2 \end{bmatrix}^T, \tag{25}$$

where $\beta = 4EI/\kappa GA{l_e}^2$.

Thus, considering the foundation and the beam, the potential and kinetic energy for an element length l_e are given by:

$$\begin{aligned} \boldsymbol{U_e} &= \frac{1}{2}\frac{2EI}{l_e}\int_{-1}^{1}\left(\frac{\partial\Psi}{\partial\xi}\right)^2 d\xi + \frac{1}{2}\frac{2\kappa GA}{l_e}\int_{-1}^{1}\left(\frac{2}{l_e}\frac{\partial V}{\partial\xi}-\Psi\right)^2 d\xi + \\ &\quad \frac{1}{2}\frac{k_f l_e}{2}\int_{-1}^{1}(V)^2\, d\xi + \frac{1}{2}\frac{2G_p}{l_e}\int_{-1}^{1}\left(\frac{\partial V}{\partial\xi}\right)^2 d\xi \end{aligned} \tag{26}$$

$$\boldsymbol{T_e} = \frac{1}{2}\frac{\rho A l_e}{2}\int_{-1}^{1}\left(\frac{\partial V}{\partial t}\right)^2 d\xi + \frac{1}{2}\frac{\rho I l_e}{2}\int_{-1}^{1}\left(\frac{\partial\Psi}{\partial t}\right)^2 d\xi. \tag{27}$$

Substituting the displacement expression, Eq. 23, into the potential energy, Eq. 27, gives:

$$\begin{aligned} \boldsymbol{U_e} &= \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{2EI}{l_e}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)']d\xi\right]\{\mathbf{v}\}_e + \\ &\quad \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{2\kappa GA}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)'-\frac{l_e}{2}\overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)'-\frac{l_e}{2}\overline{\mathbf{N}}(\xi)]d\xi\right]\{\mathbf{v}\}_e + \\ &\quad \frac{1}{2}\{\mathbf{v}\}_e^T\left[\frac{k_f l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\,d\xi + \frac{2G_p}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']\,d\xi\right]\{\mathbf{v}\}_e, \end{aligned} \tag{28}$$

where $[\mathbf{N}(\xi)'] = [\partial\mathbf{N}(\xi)/\partial\xi]$. Therefore, the element stiffness matrix is given by:

$$\begin{aligned} [\boldsymbol{k_e}] &= \left[\frac{2EI}{l_e}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)']^T[\overline{\mathbf{N}}(\xi)']d\xi + \right. \\ &\quad \frac{2\kappa GA}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)'-\frac{l_e}{2}\overline{\mathbf{N}}(\xi)]^T[\mathbf{N}(\xi)'-\frac{l_e}{2}\overline{\mathbf{N}}(\xi)]d\xi + \\ &\quad \left.\frac{k_f l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]\,d\xi + \frac{2G_p}{l_e}\int_{-1}^{1}[\mathbf{N}(\xi)']^T[\mathbf{N}(\xi)']\,d\xi\right]. \end{aligned} \tag{29}$$

Substituting the displacement expression, Eq. 23, into the kinetic energy, Eq. 27, gives:

$$\boldsymbol{T_e} = \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T\left[\frac{\rho A l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi + \frac{\rho I l_e}{2}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)]d\xi\right]\{\dot{\mathbf{v}}\}_e, \tag{30}$$

$$\begin{aligned} \boldsymbol{T_e} &= \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T\left[\frac{\rho A l_e}{2}\int_{-1}^{1}[\mathbf{N}(\xi)]^T[\mathbf{N}(\xi)]d\xi\right]\{\dot{\mathbf{v}}\}_e + \\ &\quad \frac{1}{2}\{\dot{\mathbf{v}}\}_e^T\left[\frac{\rho I l_e}{2}\int_{-1}^{1}[\overline{\mathbf{N}}(\xi)]^T[\overline{\mathbf{N}}(\xi)]d\xi\right]\{\dot{\mathbf{v}}\}_e \end{aligned} \tag{31}$$

The element mass matrix is given by:

$$[\boldsymbol{m}_e] = \left[\frac{\rho A l_e}{2} \int_{-1}^{1} [\mathbf{N}(\xi)]^T [\mathbf{N}(\xi)] d\xi + \frac{\rho I l_e}{2} \int_{-1}^{1} [\overline{\mathbf{N}}(\xi)]^T [\overline{\mathbf{N}}(\xi)] d\xi \right]. \quad (32)$$

4 NUMERICAL RESULTS

A several numerical examples were given in order to study the effect of elastic and shear layer stiffness parameters in the natural frequencies of a Timoshenko beam, as well as the effect rotational inertia and shear deformation. A simply supported uniform beam with finite length was considered, such that $L = 0.5\,m$, $E = 210\,GPa$, $G = 80.8\,GPa$, $\kappa = 5/6$, $\rho = 7850\,kg/m^3$ and $r = 0.04$.

Table 1: Comparison table for the natural frequencies of analytic and FEM analyses.

$r = 0.04$					
Mode Number	TBT	TBT (Pasternak)	FEM - 10e	FEM - 30e	FEM - 70e
1	3958.497	5209.230	5209.826	5209.295	5209.242
2	14609.403	15965.208	16000.425	15969.029	15965.908
3	29573.345	31051.689	31349.304	31084.129	31057.635
4	46937.032	48571.448	49761.199	48701.789	48595.347
5	65552.691	67371.929	70599.011	67728.235	67437.273

$r = 0.08$					
Mode Number	TBT	TBT (Pasternak)	FEM - 10e	FEM - 30e	FEM - 70e
1	7304.701	9893.118	9896.789	9893.524	9893.193
2	23468.516	26639.121	26780.991	26654.826	26642.004
3	42396.206	46341.205	47210.551	46437.646	46358.914
4	61975.547	66826.493	69604.094	67135.596	66883.247
5	81596.223	87432.265	93843.429	88151.811	87564.340

Table 1 shows the first five frequencies for two beams with different thickness on a Pasternak foundation. The second column presents the natural frequencies for a beam without foundation. The third column represents the analytic solution for a beam on Pasternak foundation, and the other columns represent FEM results for 10, 30 and 70 elements. The foundation adopted parameters were $e = 2.5$ and $p = 2.5$.

The results showed that the natural frequencies increase with the presence of the Pasternak foundation. The presence of the elastic stiffness and the shear layer increment the beam stiffness without adding mass, hence, increasing the natural frequencies. However, this increase in the natural frequency reduces drastically as the mode number rise. For higher modes, the difference between FEM and analytic solution decreases as the number of elements increase. Therefore, FEM formulation presents a high accuracy.

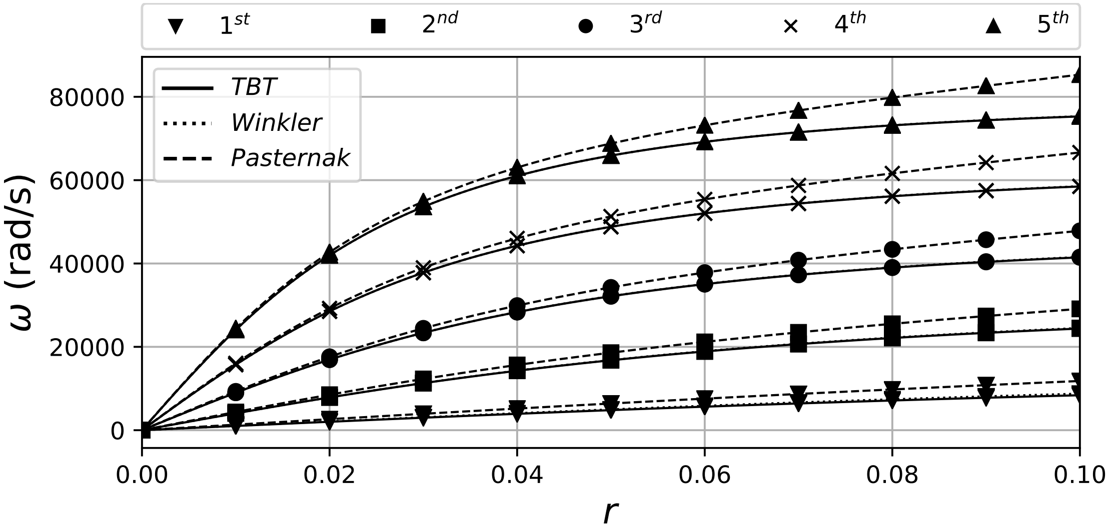

Figure 3: Frequency curves of beams on different elastic foundations (FEM - 70e).

To emphasize the effect of rotational inertia factor, the first five natural frequencies for various values of the parameter r using 70 elements are shown in Figure 3. The continuous lines corresponds a TBT ($e = 0$, $p = 0$) whereas, dotted and dashed lines corresponds to Winkler ($e = 2.5$, $p = 0$) and Pasternak ($e = 2.5$, $p = 2.5$) foundations, respectively. The figure shows that all examples increase its frequencies as the factor r increases.

For slender beams, Winkler and Pasternak foundations present a small difference in the natural frequencies. However, as the rotational inertia factor increases, the difference between them becomes greater. These behaviors on the frequencies curves highlight the influence of shear deformation for higher mode numbers and greater rotational inertia factor, hence, increasing the significance of the shear layer on the natural frequencies. Also, Winkler presents a small difference from the TBT for all frequencies.

Therefore, although Winkler increases the natural frequencies, this increase is not greater as that shown by Pasternak for non-slender beams. Considering Pasternak the analog of the TBT, which represents a more accurate theory, Winkler would correspond to Euler-Bernoulli beam theory (EBT) where its accuracy is greater for slender beams and lower mode numbers.

Figure 4 show the ratio between the frequency of Winkler foundation ($p = 0$) over a TBT ($e = 0$, $p = 0$) for various values of the parameter e for the first five frequencies. This ratio illustrates the increase in the natural frequencies for a beam on Winkler foundation with different elastic stiffness. Notice that the parameter e have a significant influence on the first frequency that becomes greater as the parameter increases. However, this increase is not notorious on the

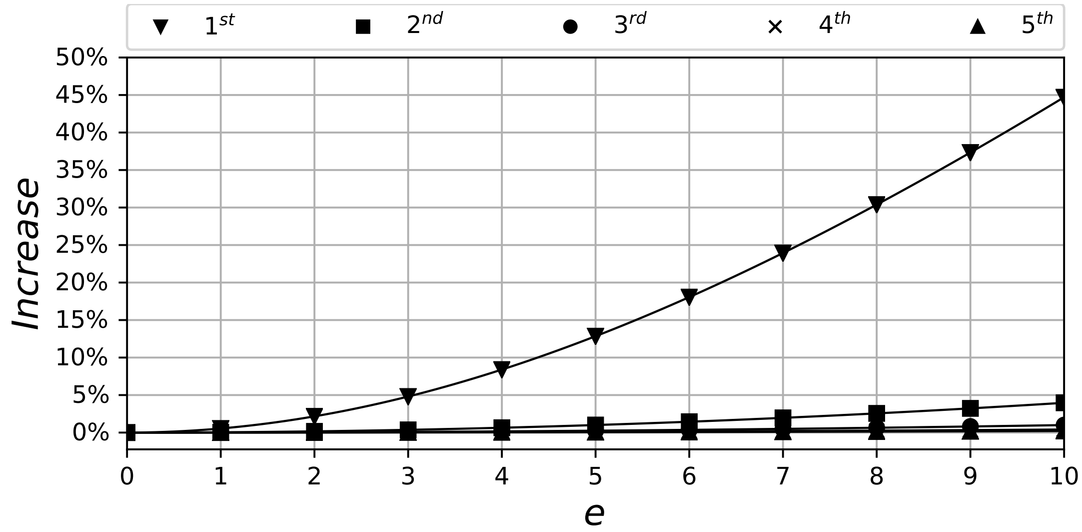

Figure 4: Increase in the natural frequencies of a beam on a Winkler ($p = 0$) foundation for various values of parameter e (FEM - 70e).

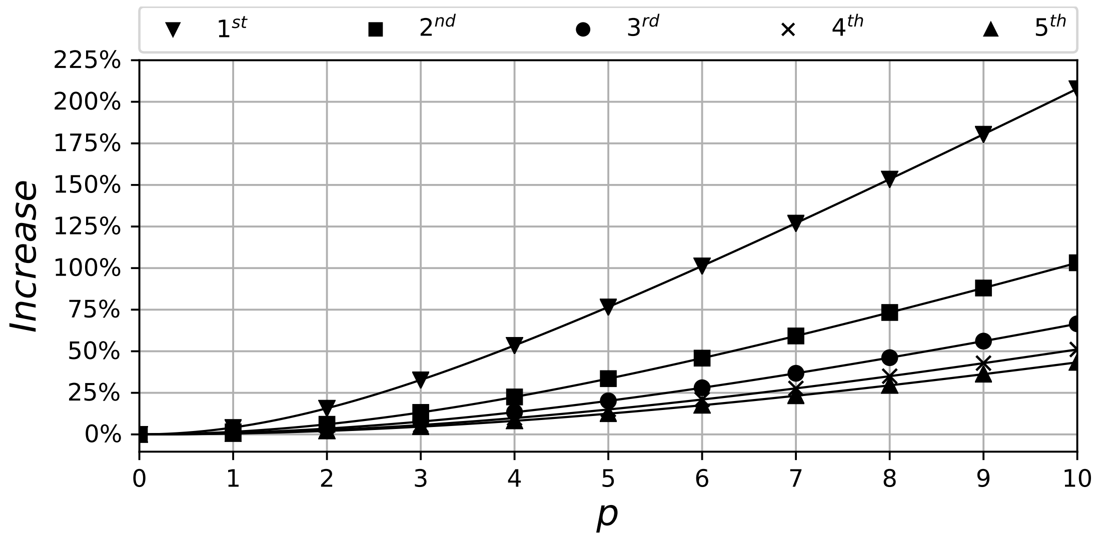

Figure 5: Increase in the natural frequencies of a beam on Pasternak ($e = 5$) foundation considering the parameter p (FEM - 70e).

other four frequencies. Therefore, as observed by Lee et al. (2014) the effect of elastic stiffness is more prominent in lower mode numbers.

In contrast to the elastic stiffness, the increase is remarkable in all modes as the shear layer stiffness is increased, as shown in Figure 5. The figure displays the ratio of the Pasternak foundation frequency over Winkler for several values of the parameter p. This ratio compares the increase of the Pasternak frequency ($e = 5$) with Winkler ($e = 5$, $p = 0$), clarifying the influence of the shear layer.

The magnitude of the increase shows that the frequency of the Pasternak foundation is mostly due the shear layer. This result indicates that the presence of shear interaction is a determinant factor not only on the displacement but also on the natural frequencies. This fact can be attested when compared with the results seen in Fig. 4, especially the curves corresponding

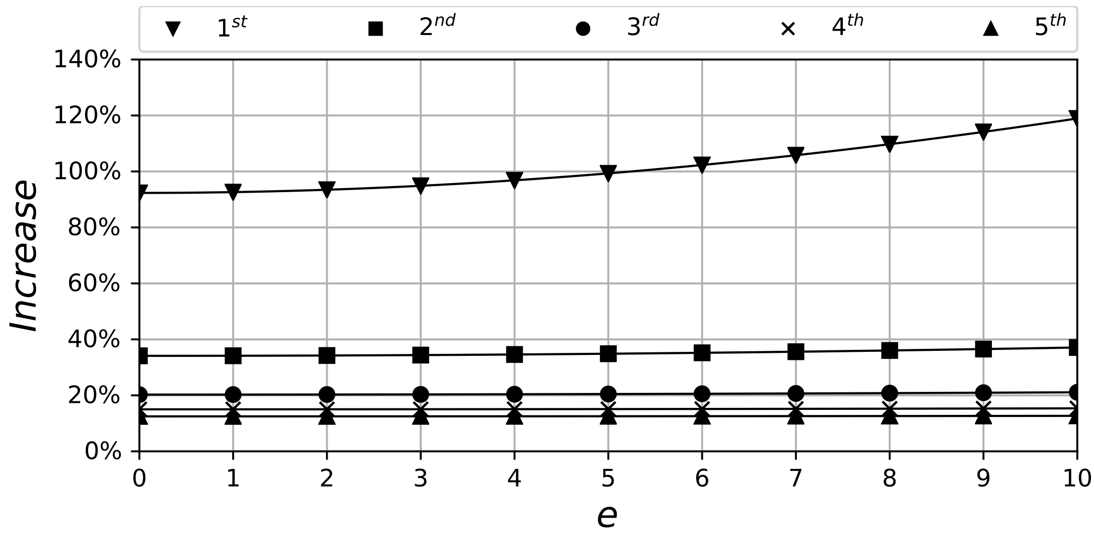

Figure 6: Increase in the natural frequencies of a beam on a Pasternak ($p = 5$) foundation for various values of e (FEM - 70e).

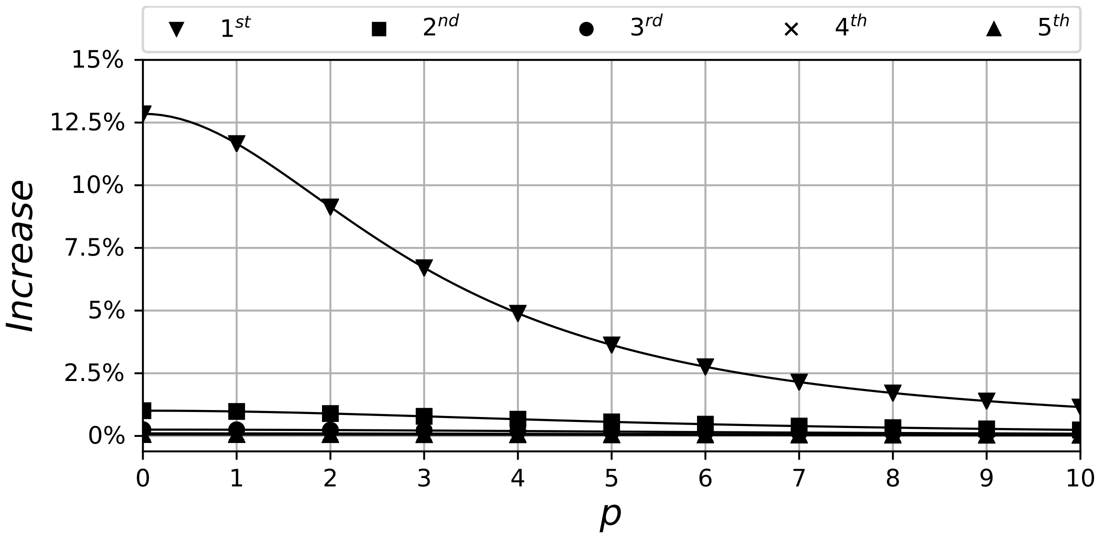

Figure 7: Influence of the foundation stiffness on a Pasternak foundation (FEM - 70e).

to 3^{rd}, 4^{th} and 5^{th} frequencies. Although the effect of the Pasternak foundation decreases for higher mode numbers, the decrement is not remarkable as supposed by Winkler foundation.

The ratio between the frequency of Pasternak ($p = 5$) foundation over a TBT ($e = 0$, $p = 0$) for various values of e for the first five frequencies is shown in Fig. 6. This ratio explicit the influence of the elastic stiffness on the frequencies of Pasternak foundation. Note that all frequencies have an increase when $e = 0$, with highlight to the first one, ratifying the major significance of the presence of a shear layer on the natural frequencies. Although all mode numbers have this initial increase, the variation in the elastic stiffness parameter presents vast changes only in the first frequency. Therefore, except in the first mode number, the foundation stiffness is almost insignificant once in the presence of shear layer. Compared with the Fig. 4, the influence of the elastic stiffness appears to be lower in the presence of the shear layer.

The effect of the elastic stiffness on the frequencies of Pasternak foundation is showed in the Figure 7. The figure presents the ratio of the Pasternak foundation frequency ($e = 5$) over an ideal foundation with only the shear layer ($e = 0$) for various values of parameter p. Observe that, as the shear layer parameter increases, the influence of the elastic stiffness decreases for all frequencies. As the first has more influence of the elastic stiffness, it is the most affected when compared to the others. Therefore, even increasing the elastic stiffness of the foundation, the shear layer will be a prevalent factor.

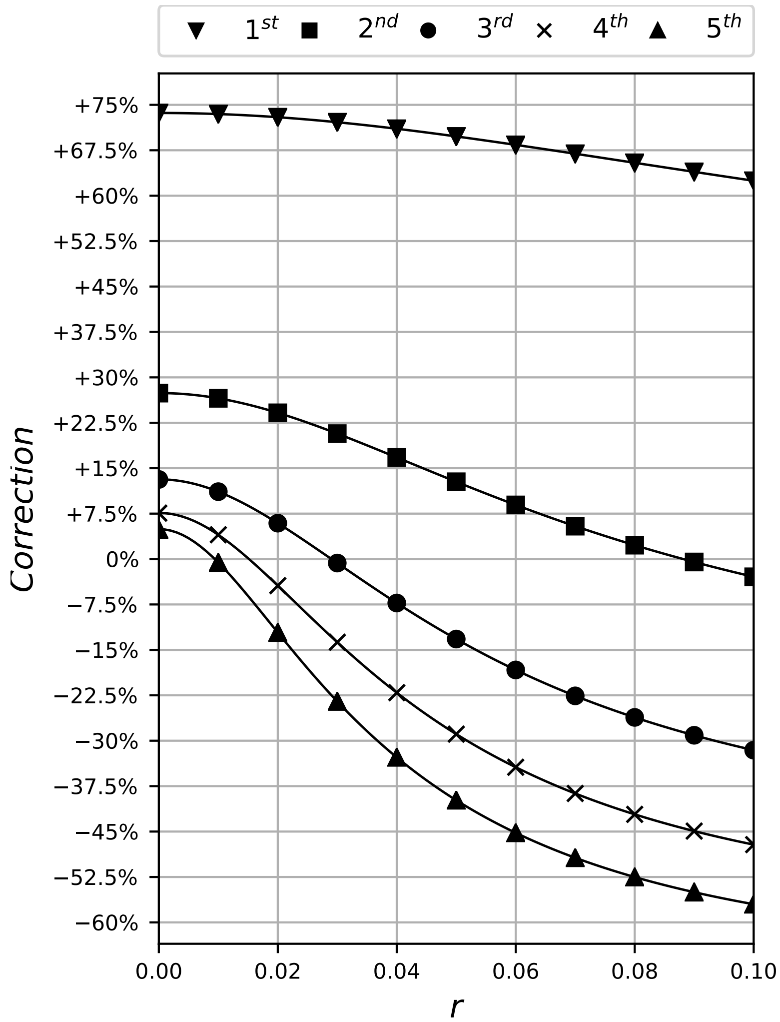

Figure 8: Correction in the natural frequencies of Euler-Bernoulli beam on Winkler foundation in comparison to Timoshenko beam on Pasternak foundation for some values of parameter r (FEM - 70e).

Figure 8 shows the relative difference between the first five frequencies of Timoshenko beam on Pasternak foundation (TP) over Euler-Bernoulli beam on Winkler foundation (EW). This relative difference indicates the correction on the natural frequencies when using highly accurate theory over simple theory as the thickness increases, in which the positive sign indicates that the TP frequency is higher than EW and a negative sign that the TP frequency is lower than EW.

For the first two modes, the frequencies of TP are much higher than EW, even for slender beams. However, for the 3^{rd}, 4^{th} and 5^{th} frequencies, TP frequencies are much lower than EW. This behavior is due to the different approach of the Winkler and Euler-Bernoulli theories. EBT overestimates the natural frequencies, while Winkler, as shown throughout this paper, underestimates. Therefore, the frequencies of EW are underestimated for slender beams and become overestimated as the thickness increases and for higher mode numbers. For specific values of rotational inertia, EW higher frequencies presents the same accuracy as TP, which explicit the transition in the behavior.

However, the first frequency does not presents this value that TP and EW presents the same accuracy and has the major correction, even for slender beams, although it has more influence of Winkler foundation and the EBT presents more accuracy for slender beams and lower modes. Therefore, this result shows the inaccuracy on using the EW for the dynamic behavior of a beam on the soil, when compared to TP.

5 CONCLUSIONS

This paper presented a finite element method for free vibration analysis of Timoshenko beams on Pasternak foundation regarding the elastic and shear layer stiffness, rotary inertia and shear deformation parameters. Cubic and quadratic polynomials were used for deflection and slope, respectively. The investigation determined that the presence of a foundation increase the natural frequencies of the beam vibration. The Winkler foundation presents a meaningful increase only for the first frequency. The Pasternak foundation presents higher frequencies than the Winkler foundation, and the increase is expressive for all modes. The growth of the frequencies on Pasternak foundation is due to the shear layer stiffness, and it reduces the influence of the elastic stiffness. Also, the use of the Euler-Bernoulli beam on Winkler foundation over the Timoshenko beam on Pasternak foundation presents a significant inaccuracy, except for specific values of rotary inertia.

REFERENCES

Azevedo, A. C., Soares, A., & Hoefel, S. S., 2016a. *The second spectrum of Timoshenko beam*. In Proceedings of the IX Congresso Nacional de Engenharia Mecânica - CONEM 2016, Fortaleza, Brazil.

Azevedo, A. C., Soares, A., & Hoefel, S. S, 2016b. *Finite element analysis of shear-deformation and rotatory inertia for beam vibration*. Revista Interdisciplinar de Pesquisa em Engenharia - RIPE, vol. 2, n. 34, pp. 86–103.

Chen, W. Q., Lü, C. F., & Bian, Z. G., 2004. *A mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation*. Applied Mathematical Modelling, vol. 28, pp. 877–890.

De Rosa, M. A., 1995. *Free vibrations of Timoshenko beams*. Computers and Structures, vol. 57, n. 1, pp. 151–156.

Dutta, S. C. & Roy, R., 2002. *A critical review on idealization and modeling for interaction among soilfoundationstructure system*. Computers and Structures, vol. 80, pp. 1579–1594.

El-Mously, M., 1999. *Fundamental frequencies of Timoshenko beams mounted on Pasternak foundation*. Journal of Sound and Vibration, vol. 228, n. 2, pp. 452–457.

Ghannadiasl, A. & Mofid, M., 2015. *An Analytical Solution for Free Vibration of Elastically Restrained Timoshenko Beam on an Arbitrary Variable Winkler Foundation and Under Axial Load*. Latin American Journal of Solids and Structures, vol. 12, pp. 2417–2438.

Hetenyi, M., 1946. *Beams on Elastic Foundation*. The University of Michigan Press.

Kerr, A. D., 1964. *Elastic and Viscoelastic Foundation Models*. Journal of Applied Mechanics, pp. 491–498.

Lee, J. K., Jeong, S., & Lee, J, 2014. *Natural frequencies for flexural and torsional vibrations of beams on Pasternak foundation*. Soils and Foundations, vol. 54, n. 6, pp. 1202–1211.

Pasternak, P.L., 1954. *On a New Method of Analysis of an Elastic Foundation by Means of two Foundation Constants (in Russian)*. Gosudarstvenno Izdatefslvo Literaturi po Stroitelstvu i Arkhitekture, vol. 21, Moscow, U.S.S.R.

Selvadurai, A. P. S., 1979. *Elastic Analysis of Soil-foundation Interaction*. Elsevier Science Ltd.

Soares, A., & Hoefel, S. S., 2015. *Modal analysis for free vibration of four beam theories*. In Proceedings of the 23rd International Congress of Mechanical Engineering, Rio de Janeiro, Brazil.

Thambiratnam, D. & Zhuge, Y., 1996. *Free vibration analysis of beams on elastic foundation*. Computers and Structures, vol. 60, n. 6, pp. 971–980.

Timoshenko, S. P., 1921. *On the correction for shear of the differential equation for transverse vibration of prismatic bars*. Philosophical Magazine, vol. 41, pp. 744-746.

Wang, T. M. & Stephens, J. E., 1977. *Natural frequencies of Timoshenko beams on Pasternak foundations*. Journal of Sound and Vibration, vol. 51, pp. 149–155.

Winkler, E., 1867. *Die Lehre Von Der Elasticitaet und Festigkeit*.

Yokoyama, T., 1987. *Vibrations and transient responses of Timoshenko beams resting on elastic foundations*. Ingenieur-Archiv, vol. 57, pp. 81–90.

Yokoyama, T., 1996. *Vibration analysis of Timoshenko beam-columns on two-parameter elastic foundations*. Computers and Structures, vol. 61, n. 6, pp. 995–1007.